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Geometric singular perturbation analysis of oxidation heat pulses for two-phase flow in porous media*

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— Dedicated to Constantine Dafermos on his 60th birthday

Abstract. When air or oxygen is injected into a petroleum reservoir, and oxidation or combustion is induced, a combustion front forms if heat loss to the surrounding rock formation is negligible. Here, we employ a simple model for combustion, which takes into account oil viscosity reduction, but neglects gas density dependence on temperature and uses a simplified oxidation reaction. We show that for small heat loss, this combustion front is actually the lead part of a pulse, while the trailing part of the pulse is a slow cooling process. If the heat loss is too large, we show that such a pulse does not exist. The proofs use geometric singular perturbation theory and center manifold reduction.

Keywords: combustion, porous medium, multiphase flow, conservation laws.

1 Introduction

In situ combustion is a method of oil recovery that uses a chemical reaction to cause a temperature increase; among other effects, oil viscosity is reduced and the oil flows more readily. It has been successfully used in many oil fields in many countries, especially in the former Soviet Union, and has been the subject of a number of papers in the petroleum engineering literature [2],[3],[5],[7], [14],[18], and a few papers in the mathematical literature [8],[10],[12]. The mathematical

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theory of *in situ* combustion is an aspect of the theory of combustion in multiphase flow in a porous medium. The latter subject is little developed, especially by comparison to the theory of combustion in gases.

In this work, we study a very simplified model, which is more representative of low temperature oxidation than *in situ* combustion [17]. We find conditions under which there exists a traveling exothermic oxidation pulse. These conditions are strict inequalities. We show that such a pulse has a triangular shape. Its lead part is a steep front where the reaction takes place and the temperature quickly increases. Its trailing part has a mild slope; the main process in this part of the wave is cooling by heat loss to the overburden rock formation. Such a pulse was observed numerically in the work of Crookston *et al.*; see Fig. 1 in [7].

We have simplified substantially the equations used in petroleum engineering [7], [5]. We assume that the only component in the gaseous phase present initially is oxygen, and that the only gaseous product of the chemical reaction is carbon dioxide. We assume chemical reactions with a single reaction rate for the several hydrocarbons contained in the oil, instead of the multiple reaction rates that actually occur during *in situ* combustion.

Most significantly, we neglect mass transfer from the oleic to the gaseous phase due to the reaction, and we neglect the effect of temperature change on the gaseous phase density. These simplifications greatly facilitate the analysis: the total seepage velocity becomes constant (just as in the classical Buckley-Leverett treatment) and can be factored out of the flow solutions.

We also make simplifications that are more justifiable. First, we do not allow the presence of liquid water, which is reasonable if the temperature is relatively high, and if there was no liquid water initially. Thus we treat a two-phase flow (oleic and gaseous), and avoid difficulties associated with the analysis of steam production and condensation in the reservoir, as well as those associated with three-phase flow. These difficulties should be irrelevant to the key aspect of both low temperature oxidation and *in situ* combustion, the reduction of oil viscosity by temperature increase. In work on oil recovery by steam injection, for example, similar effects occur in both two-phase and three-phase models [4].

Second, we neglect the solubility of the reaction product carbon dioxide in oil, which occurs on a slow time scale. This is valid if the seepage velocity is large enough.

Our analysis is motivated by the following physical situation. Initially, there is a uniform distribution of oleic and gaseous phases in a porous rock. The fluids are being displaced as a whole by proper injection at the left end. At a certain time and location, ignition starts. Is it possible that an oxidation pulse forms and propagates as a traveling wave?

We find that if the heat loss to the overburden is too large, no such oxidation pulse can exist. On the other hand, if the heat loss is small enough, a traveling oxidation pulse indeed exists as a solution to the equations. Proving the existence of such a pulse is an important step toward solving the ignition problem for flow in a porous medium.

A more realistic, and more complicated, model is considered in [17]. Much of the analysis is similar to the analysis in the present paper, and analogous results hold.

We now preview the rest of the paper. In Secs. 2 and 3, we explain the system of conservation laws that we shall study. The system contains source terms due to the chemical reaction. The model has two phases, oleic and gaseous, and three components: the gaseous phase is divided into oxygen and carbon dioxide. A more detailed explanation of the model can be found in [9]. We have added to the model of [9] a term that represents heat loss to the overburden rock.

In Sec. 4, we derive the ordinary differential equations for traveling waves. Then, motivated by geometric singular perturbation theory, which is frequently used in the study of traveling waves [15], we first study the reduced system of equations in which heat loss vanishes. This system is the one studied in [9]. We focus our attention on two curves of equilibria of the reduced system, one consisting of equilibria at which the temperature is that of the surrounding rock and the percentage of gas that has burned varies, the other consisting of equilibria at which the temperature is above that of the surrounding rock and all of the gas has burned. These curves of equilibria of the reduced system are studied in Sec. 5, and the associated invariant manifolds are found in Sec. 6. The curve of high-temperature equilibria is normally hyperbolic; the curve of low-temperature equilibria is not. For small heat loss, the low-temperature equilibria remain equilibria; the high-temperature equilibria do not.

In Sec. 7, we state precisely the main result of this paper, which asserts the existence of certain connecting orbits for the ordinary differential equation with small heat loss. The speed of the traveling wave is approximately the speed for which the reduced system has a connection with a special structure between certain high- and low-temperature equilibria: it is a connection between the unstable manifold of a hyperbolic equilibrium and the stable manifold of a non-hyperbolic equilibrium, rather than a connection that arrives at the nonhyperbolic equilibrium tangent to its center direction. The traveling wave with small heat loss is approximately this connection followed by slow drift along the curve of high-temperature equilibria of the reduced system. The temperature gradually falls along this curve of equilibria until it meets the curve of low-temperature equilibria. The traveling wave terminates near this point of intersection, which is a further degeneracy.

In Sec. 8 we reinterpret the results of Secs. 5 and Sec. 6 to make them more useful in studying the ordinary differential equation with small heat loss, which is treated as a perturbation term. For small heat loss, the front end of the traveling pulse, which is dominated by heat generation, is studied in Sec. 9, and the back end, which is dominated by heat loss, in Secs. 10 and 11. The latter section treats the termination of the traveling wave using center manifold reduction at the intersection of the two curves of equilibria of the reduced system.

The proof of the existence result is completed in Sec. 12. In Sec. 13, we show that large heat loss prevents the existence of a traveling pulse. In Section 14, conclusions and discussion are presented. An appendix summarizes nomenclature used throughout the paper.

2 The oxidation model with heat loss

We will study a system of reaction-convection-diffusion equations, which models oxidation in a one-dimensional petroleum reservoir [9]:

$$s_t + f(s,\theta)_x = \big(h(s,\theta)s_x\big)_x, \tag{2.1}$$

$$\begin{aligned}\big((\alpha - s)\theta - \eta\epsilon s\big)_t + \Big(\big(\beta - f(s,\theta)\big)\theta - \eta\epsilon f(s,\theta)\Big)_x \\ = -\big((\theta+\eta\epsilon)h(s,\theta)s_x\big)_x + \gamma\theta_{xx} - \delta(\theta-\theta_0),\end{aligned} \tag{2.2}$$

$$(\epsilon s)_t + \big(\epsilon f(s,\theta)\big)_x = \big(\epsilon h(s,\theta)s_x\big)_x + \zeta s q(\theta,\epsilon). \tag{2.3}$$

The reservoir contains a gaseous phase, consisting of a mixture of oxygen and carbon dioxide, and an oleic phase in the pores of a rock matrix: s is the gaseous phase saturation, so that $1 - s$ is oil saturation. The temperature is θ. The fraction of initial oxygen that has burnt (converted to carbon dioxide) is ϵ, so that $1-\epsilon$ is the fraction that has not burnt. It is assumed that the oil mass loss is negligible. The variables in (2.1)–(2.3) are s, θ, and ϵ; δ is a parameter; θ_0, α, β, γ, η, and ζ are positive constants. The functions f and h are discussed in the next section. Eq. (2.1) expresses conservation of mass of the gaseous and oleic phases, combined with Darcy's law of force. Eq. (2.2) expresses conservation of energy. Eq. (2.3) describes the chemical reaction.

The temperature of the surrounding rock formation is θ_0. The last term in Eq. (2.2) represents heat loss from the multiphase fluid to the rock formation according to Newton's Law of Cooling.

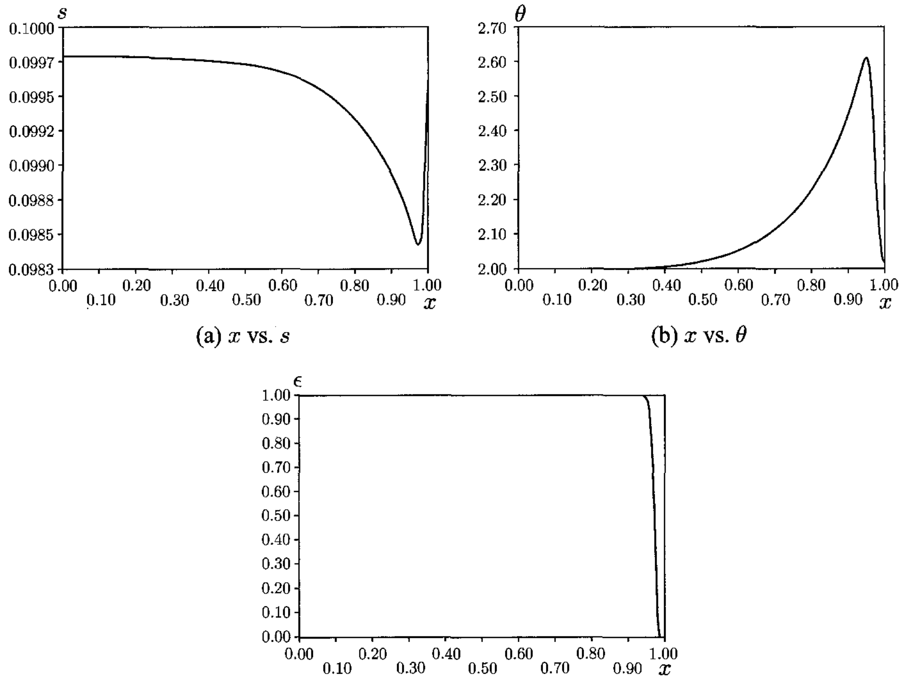
The result of this paper is that, under certain assumptions, the system (2.1)–(2.3) admits, for small $\delta > 0$, a traveling wave solution $\big(s(z), \theta(z), \epsilon(z)\big)$, $z = x - \sigma t$, that represents an oxidation front followed by a slow cooling process. As stated in the Introduction, the oxidation front is a solution of the system (2.1)–(2.3) with $\delta = 0$ and was studied in [9]. The slow cooling behind the oxidation front is due to the inclusion of the last term in Eq. (2.2). The speed σ of the traveling wave is $\sigma(\delta) = \sigma_0 + O(\delta)$, where $\sigma_0 > 0$ is the speed for which the oxidation front when $\delta = 0$ corresponds to a connection between the unstable manifold of a certain hyperbolic equilibrium and the stable manifold of a certain nonhyperbolic equilibrium, rather than a connection that arrives at the nonhyperbolic equilibrium tangent to its center direction. When $\delta = 0$, the latter connections exist for an open interval of σ. However, they do not persist when $\delta > 0$, because the 0 eigenvalue at the nonhyperbolic equilibrium becomes positive. One assumption that we make is that, when $\delta = 0$, the connection between the unstable manifold of the hyperbolic equilibrium and the stable manifold of the nonhyperbolic equilibrium that exists when $\sigma = \sigma_0$ breaks in a nondegenerate manner as σ varies.

A numerically computed oxidation pulse for our model with $\delta > 0$ is shown in Fig. 2.1. The wave is shown for fixed t. Since $\sigma > 0$, the wave moves to the right. At the oxidation front, ϵ falls rapidly from near 1 (burnt) to 0 (unburnt). Also at the oxidation front, the values of s and θ change rapidly from values associated with the oxidation process to their values s_0 and θ_0 in the surrounding rock formation. Behind the oxidation front, s and θ return more slowly to s_0 and θ_0. The limiting value of ϵ at the left is $\epsilon^- = 1 - O(e^{-\frac{k}{\delta}})$ for some positive constant k. Thus the loss of heat to the surrounding rock formation prevents the oxygen from being completely consumed, a fact with important practical consequences. However, this effect is not visible in the simulations we have done.

3 Explanation of Equations

In this section we explain the terms in Eqs. (2.1)–(2.3) in more detail.

The relative permeabilities of the gaseous and oleic phases, k_g and k_o, are dimensionless functions of gaseous phase saturation s and oleic phase saturation $1 - s$ respectively. The viscosities μ_g and μ_o of the gaseous and oleic phases are functions of the temperature θ. The relative mobilities of the gaseous and oleic phases, λ_g and λ_o, are functions of the saturations s of the gaseous phase



(a) x vs. s (b) x vs. θ (c) x vs. ϵ

Figure 2.1: The traveling wave at a fixed time: s, θ, and ϵ are shown as functions of x. An interval of length 750 on the x-axis is shown, scaled to the interval [0,1]. The flux function is $f(s,\theta) = \frac{s^2}{s^2+(1+0.1\theta)(1-s)^2}$, and $h(s,\theta) = -1$. Parameter values are $\alpha = 3.0$, $\beta = 1.2$, $\gamma = 1.0$, $\eta = 5.0$, and $\zeta = 1.0$. The value of δ is .003334541, for which the corresponding wave speed is $\sigma = .5105858$. The traveling wave was computed using AUTO [11].

and $1-s$ of the oleic phase respectively, and of the temperature θ:

$$\lambda_g(s,\theta) = \frac{k_g(s)}{\mu_g(\theta)} \quad \text{and} \quad \lambda_o(1-s,\theta) = \frac{k_o(1-s)}{\mu_o(\theta)}. \tag{3.1}$$

The "fractional flow function" of the gaseous phase $f(s,\theta)$ is then

$$f(s,\theta) = \frac{\lambda_g(s,\theta)}{\lambda_g(s,\theta)+\lambda_o(1-s,\theta)}. \tag{3.2}$$

The pressures in the gaseous and oleic phases, p_g and p_o, are functions of s and $1-s$ respectively. The capillary pressure p_c is a decreasing function of s

measured in the laboratory and defined by

$$p_c(s) = p_o(1-s) - p_g(s). \tag{3.3}$$

Then we define the function

$$h(s,\theta) = -\lambda_o(s,\theta) f(s,\theta) p_c'(s). \tag{3.4}$$

We assume $p_c'(s) < 0$ for $0 < s < 1$. Thus

$$h(s,\theta) > 0 \text{ for } 0 < s < 1. \tag{3.5}$$

Let ρ_g, ρ_o, and ρ_r denote the densities of gaseous, oleic, and rock phases; let C_g, C_o, and C_r denote the heat capacities of gaseous, oleic, and rock phases; let ϕ denote the rock porosity, the fraction of total volume occupied by the fluid phases; and let K denote the absolute permeability of the rock, the porous medium's capability of allowing fluid flow. Let Q denote the heat released by the oxidation per unit mass. We will assume that these are all constants.

The thermal conductivities of the gaseous, oleic, and rock phases in the x-direction are all assumed to equal a constant κ; we make this unphysical assumption to facilitate the analysis. Thermal conductivity transverse to the x-direction is assumed to be a constant κ_l.

We assume that there is incompressible flow of the gaseous and oleic phases. The total seepage flow of both gaseous and oleic phases v is then a function of time only, determined by the boundary conditions. For simplicity we assume that v is constant.

The quantities α, β, γ, η, ζ, and δ are defined by

$$\alpha = \frac{\rho_o C_o + \rho_r C_r/\phi}{\rho_o C_o - \rho_g C_g}, \quad \beta = \frac{\rho_o C_o}{\rho_o C_o - \rho_g C_g}, \quad \gamma = \frac{\phi \kappa}{K(\rho_o C_o - \rho_g C_g)},$$
$$\eta = \frac{\rho_g Q}{\rho_o C_o - \rho_g C_g}, \quad \zeta = \frac{\phi K}{v^2}, \quad \delta = \frac{\kappa_l}{v}. \tag{3.6}$$

The first five are positive constants; the last will be regarded as a nonnegative parameter.

The function $q(\theta, \epsilon)$ denotes the volumetric fraction of burnt gaseous phase generated per unit time.

All these terms were defined, and the equations derived, in [8] and [9], except that κ_l was not defined there, and the last term in Eq. (2.2) was omitted.

The function $q(\theta, \epsilon)$ will be assumed to have the following form (see Eq. 95.02 of [6]), a version of Arrhenius's law:

$$q(\theta, \epsilon) = \begin{cases} K_\infty(1-\epsilon)e^{-\frac{L}{\theta-\theta_0}} & \text{if } \theta > \theta_0, \\ 0 & \text{if } \theta \le \theta_0. \end{cases} \tag{3.7}$$

For simplicity we shall take $K_\infty = L = 1$.

We assume that $f(s, \theta)$ is C^2 and S-shaped in s for each θ. See Figure 3.1. More precisely, we assume that $f(0, \theta) = 0$, $f(1, \theta) = 1$, and, for each θ, $f_{ss}(s, \theta)$ is first positive and then negative for $0 < s < 1$. We further assume that $f_\theta < 0$. These assumptions are used to model two-phase thermal flow in a porous medium, for which oil viscosity is a decreasing function of temperature. They hold, for example, for $k_g(s) = s^2$ and $k_o(1-s) = (1-s)^2$, which were used in the computation that produced Figure 2.1.

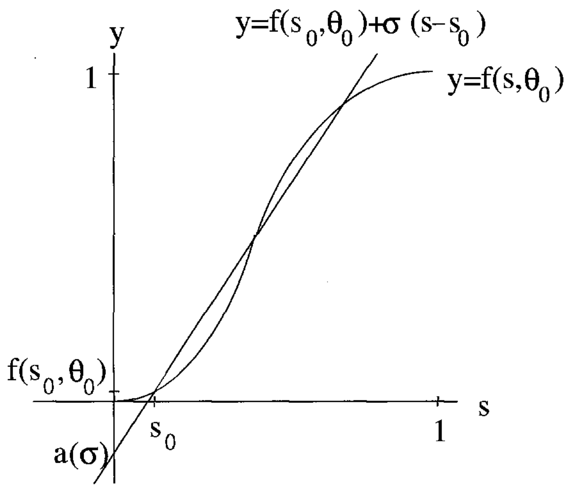


Figure 3.1: Graphs of $y = f(s, \theta_0)$ and $y = f(s_0, \theta_0) + \sigma(s - s_0)$.

4 The Traveling Wave System

We shall look for traveling wave solutions $\big(s(z), \theta(z), \epsilon(z)\big)$ of the system (2.1)–(2.3), with $z = x - \sigma t$.

The traveling wave solutions will be required to approach constant limits as

$z \to \pm\infty$. Thus we define

$$\mathcal{X} = \{g : \Re \to \Re : g \text{ is } C^1, \lim_{z\to\pm\infty} g(z) \text{ exists, and } \lim_{z\to\pm\infty} \frac{dg}{dz} = 0\}.$$

Let $\mathcal{X}^n = \mathcal{X} \times \ldots \times \mathcal{X}$ (n times).

Given (s, θ, ϵ) in $\mathcal{X}^3$, let

$$s^\pm = \lim_{z\to\pm\infty} s(z), \qquad \theta^\pm = \lim_{z\to\pm\infty} \theta(z), \qquad \epsilon^\pm = \lim_{z\to\pm\infty} \epsilon(z).$$

Recall that s_0 and θ_0 are the gaseous saturation and temperature of the surrounding reservoir. We assume $0 < s_0 < 1$. We shall look for solutions $\big(s(z), \theta(z), \epsilon(z)\big)$ of the system (2.1)–(2.3) in $\mathcal{X}^3$ with

$$s^+ = s_0, \quad \theta^\pm = \theta_0, \quad \epsilon^+ = 0. \tag{4.1}$$

The values of s^- and ϵ^- must be determined. On physical grounds it is natural to expect that $s^- = s_0$ and $\epsilon^- \approx 1$.

We define

$$a(\sigma) = \sigma s_0 - f(s_0, \theta_0), \tag{4.2}$$

$$b(\sigma) = -\beta + \sigma\alpha - a(\sigma). \tag{4.3}$$

We shall sometimes suppress the dependence of a and b on the parameter σ.

In Eqs. (2.1)–(2.3) we let $(s, \theta, \epsilon) = \big(s(z), \theta(z), \epsilon(z)\big)$, $z = x - \sigma t$. Eq. (2.1) can then be integrated. Using the boundary conditions $s(\infty) = s_0$, $s'(\infty) = 0$, and $\theta(\infty) = \theta_0$ yields

$$\frac{ds}{dz} = \frac{a - \sigma s + f(s, \theta)}{h(s, \theta)}. \tag{4.4}$$

Substitution of Eq. (4.4) into Eq. (2.2) and Eq. (2.3) yields

$$\frac{d^2\theta}{dz^2} = \frac{1}{\gamma}\frac{d}{dz}(-b\theta + \eta\epsilon a) + \frac{\delta}{\gamma}(\theta - \theta_0) \tag{4.5}$$

and

$$\frac{d\epsilon}{dz} = -\frac{\zeta}{a} s q(\theta, \epsilon). \tag{4.6}$$

Proposition 4.1. *Let* $\big(s(z), \theta(z), \epsilon(z)\big) \in \mathcal{X}^3$ *with* $s^+ = s_0$, $\theta^+ = \theta_0$, $\epsilon^+ = 0$. *Then the following are equivalent.*

(1) $\big(s(z), \theta(z), \epsilon(z)\big)$ *satisfies Eqs. (4.4)–(4.6).*

(2) There is a function $\Psi(z) \in \mathcal{X}$, *with* $\lim_{z\to\infty} \Psi(z) = 0$, *such that* $\big(s(z), \theta(z), \epsilon(z), \Psi(z)\big)$ *satisfies the system consisting of Eq. (4.4),*

$$\frac{d\theta}{dz} = \frac{1}{\gamma}\Big(- b(\theta - \theta_0) + \eta\epsilon a + \Psi\Big), \tag{4.7}$$

Eq. (4.6), and

$$\frac{d\Psi}{dz} = \delta(\theta - \theta_0). \tag{4.8}$$

Proof. To see that (2) implies (1), we only need to check that $\theta(z)$ satisfies Eq. (4.5). Just differentiate Eq. (4.7) and with respect to z and use Eq. (4.8).

To prove that (1) implies (2), let $\big(s(z), \theta(z), \epsilon(z)\big)$ satisfy (1). We multiply Eq. (4.5) by -1 and integrate from z to ∞. We obtain

$$\frac{d\theta}{dz} = \frac{1}{\gamma}\Big(- b(\theta(z) - \theta_0) + \eta\epsilon(z)a - \delta \int_z^\infty (\theta(\tau) - \theta_0)\, d\tau\Big). \tag{4.9}$$

For $\delta \neq 0$, the integral is finite because the other terms are. Hence we can define

$$\Psi(z) = -\delta \int_z^\infty (\theta(\tau) - \theta_0) d\tau. \tag{4.10}$$

(For $\delta = 0$ we just set $\Psi(z) = 0$ for all z.) Then $\big(s(z), \theta(z), \epsilon(z), \Psi(z)\big)$ satisfies (2). □

Motivated by Proposition 4.1, we shall study the first-order system consisting of Eqs. (4.4), (4.7), (4.6), and (4.8), which we shall call the *Traveling Wave System*. We shall look for a solution $\big(s(z), \theta(z), \epsilon(z), \Psi(z)\big)$ that is in $\mathcal{X}^4$, and that satisfies the boundary conditions (4.1) and the additional boundary condition $\lim_{z\to\infty} \Psi(z) = 0$. The values of s^-, ϵ^-, and Ψ^- must be determined.

The Traveling Wave System has σ and δ as parameters. Recall that a and b are functions of σ. The parameter δ appears only in the fourth equation. Moreover, when δ=0, $\frac{d\Psi}{dz} = 0$, so Ψ is constant.

To bring out this structure, let

$$u = (s, \theta, \epsilon), \quad w = (u, \Psi) = (s, \theta, \epsilon, \Psi).$$

We write the Traveling Wave System as

$$\frac{dw}{dz} = F(w, \sigma, \delta), \tag{4.11}$$

with parameters σ and δ. Eq. (4.11) can also be written

$$\frac{du}{dz} = G(u, \Psi, \sigma), \tag{4.12}$$

$$\frac{d\Psi}{dz} = \delta(\theta - \theta_0). \tag{4.13}$$

We can think of (4.12) as defining a system of ODEs on (s, θ, ϵ)-space with parameters Ψ and σ.

5 Equilibria of $\frac{du}{dz} = G(u, \Psi, \sigma)$

In this section we shall determine the equilibria of $\frac{du}{dz} = G(u, \Psi, \sigma)$. We first define three sets. The definitions use assumptions (I1)–(I4) to be given shortly.

$$I = \{\sigma : \text{conditions (I1)–(I4) are satisfied}\},$$
$$J = \{(\Psi, \sigma) : -\eta a(\sigma) \leq \Psi \leq 0,\ \sigma \in I\},$$
$$S = \{(s, \theta, \epsilon, \Psi, \sigma) : 0 < s < 1,\ \theta_0 \leq \theta,\ 0 \leq \epsilon \leq 1,\ (\Psi, \sigma) \in J\}.$$

We shall restrict our attention to equilibria in S.

Equilibria of $\frac{du}{dz} = G(u, \Psi, \sigma)$ in S satisfy

$$a - \sigma s + f(s, \theta) = 0, \tag{5.1}$$

$$-b(\theta - \theta_0) + \eta\epsilon a + \Psi = 0, \tag{5.2}$$

$$q(\theta, \epsilon) = 0. \tag{5.3}$$

From Eq. (5.3), $\theta = \theta_0$ or $\epsilon = 1$.

1. Equilibria with $\theta = \theta_0$. Substituting $\theta = \theta_0$ in Eq. (5.2) yields

$$\epsilon = -\frac{\Psi}{\eta a}. \tag{5.4}$$

Substituting $\theta = \theta_0$ and $a = \sigma s_0 - f(s_0, \theta_0)$ in Eq. (5.1) yields

$$f(s, \theta_0) - f(s_0, \theta_0) = \sigma(s - s_0).$$

Figure 3.1 shows the curve $y = f(s, \theta_0)$ and the line $y = f(s_0, \theta_0) + \sigma(s - s_0)$. The line meets the y-axis at $y = -a(\sigma)$.

Since by definition the particle speed in a conservation law $s_t + f_x = 0$ is f/s, the positive or negative sign of $a(\sigma)$ corresponds to wave speeds σ larger or smaller than particle speed. In this work, we will consider *forward-moving oxidation waves*, which move faster than gas particle velocity. They are the waves of interest when the oxidation starts at the well where oxygen is injected. More precisely, we assume

(I1) $a(\sigma) > 0$.

The curve and the line meet at $s = s_0$ and possibly at other points. We shall assume

(I2) $\sigma - f_s(s_0, \theta_0) > 0$.

This assumption implies that $\sigma > 0$. It says that the saturation wave characteristic speed ahead of the oxidation wave is slower than the oxidation wave itself. This is Lax's classical condition for the traveling wave to give rise to a shock [19] in the zero diffusion limit.

We define the following equilibria of $\frac{du}{dz} = G(u, \Psi, \sigma)$ with $(\Psi, \sigma) \in J$:

$$m(\Psi, \sigma) = (s, \theta, \epsilon) \text{ where } s = s_0, \ \theta = \theta_0, \ \text{and } \epsilon = -\frac{\Psi}{\eta a(\sigma)}.$$

As Ψ increases from $-\eta a(\sigma)$ to 0, the ϵ-coordinate of $m(\Psi, \sigma)$ decreases linearly from 1 to 0.

2. Equilibria with $\epsilon = 1$.

We assume that the thermal wave characteristic speed ahead of the oxidation wave is slower than the oxidation wave itself. That is:

(I3) $b(\sigma) > 0$.

Together with (I2), condition (I3) is part of the Lax condition for the oxidation wave to become a 2-shock in the zero diffusion limit.

Substituting $\epsilon = 1$ in Eq. (5.2) and using the definition of b, we obtain

$$\theta = \theta_0 + \frac{1}{b}(\Psi + \eta a). \tag{5.5}$$

From Eq. (5.1),

$$f(s, \theta) - f(s_0, \theta_0) = \sigma(s - s_0). \tag{5.6}$$

By (I2) and the Implicit Function Theorem, Eq. (5.6) can be solved for s in terms of θ and σ near any point (s_0, θ_0, σ), $\sigma \in I$. More precisely, we shall need the following: there is a number θ^* and a function $s_0(\theta, \sigma)$, defined for θ near θ_0 and $\sigma \in I$, such that $s_0(\theta_0, \sigma) = s_0$, and for $\theta_0 \leq \theta < \theta^*$,

$$f\big(s_0(\theta, \sigma), \theta\big) - f(s_0, \theta_0) = \sigma\big(s_0(\theta, \sigma) - s_0\big)$$

and $\sigma - f_s(s_0(\theta, \sigma), \theta) > 0$.

Let $\theta_1(\sigma) = \theta_0 + \frac{\eta a(\sigma)}{b(\sigma)}$; notice that if $\theta = \theta_1(\sigma)$ in Eq. (5.5), then $\Psi = 0$. We shall assume:

(I4) $\theta_1(\sigma) < \theta^*$.

Then for $(\Psi, \sigma) \in J$, there is an equilibrium of $\frac{du}{dz} = G(u, \Psi, \sigma)$ with $\epsilon = 1$ at the point

$$n(\Psi, \sigma) = (s, \theta, \epsilon) \text{ where } s = s_0(\theta, \sigma),\ \theta = \theta_0 + \frac{1}{b(\sigma)}\big(\Psi + \eta a(\sigma)\big),\ \epsilon = 1.$$

As Ψ increases from $-\eta a(\sigma)$ to 0, the θ-coordinate of $n(\Psi, \sigma)$ increases linearly from θ_0 to $\theta_1(\sigma)$.

The equilibria $m(\Psi, \sigma)$ and $n(\Psi, \sigma)$ are sketched in Figure 5.1. Notice that $m(\Psi, \sigma) = n(\Psi, \sigma)$ at $\Psi = -\eta a(\sigma)$.

For fixed Ψ and σ, the linearization of $\dfrac{du}{dz} = G(u, \Psi, \sigma)$ at an equilibrium is given by the matrix

$$M = \begin{pmatrix} \frac{f_s-\sigma}{h} & \frac{f_\theta}{h} & 0 \\ 0 & -\frac{b}{\gamma} & \frac{\eta a}{\gamma} \\ -\frac{\zeta}{a}q & -\frac{\zeta}{a}sq_\theta & -\frac{\zeta}{a}sq_\epsilon \end{pmatrix}.$$

(See Eq. 5.6 of [8].) At the equilibria $m(\Psi, \sigma)$, where $\theta = \theta_0$, we have $q = q_\theta = q_\epsilon = 0$, so the eigenvalues are $\frac{f_s-\sigma}{h}$, $-\frac{b}{\gamma}$, and 0. The first two of these eigenvalues are negative. At the equilibria $n(\Psi, \sigma)$, where $\epsilon = 1$, we have $q = q_\theta = 0$, so the eigenvalues are $\frac{f_s-\sigma}{h}$, $-\frac{b}{\gamma}$, and $-\frac{\zeta}{a}sq_\epsilon$. The first two are negative. The last is positive for $\Psi > -\eta a(\sigma)$. When $\Psi = -\eta a(\sigma)$, $m(\Psi, \sigma)$ and $n(\Psi, \sigma)$ coincide, and the positive eigenvalue at $n(\Psi, \sigma)$ becomes 0.

6 Invariant manifolds of $\frac{du}{dz} = G(u, \Psi, \sigma)$

We now consider invariant manifolds of $\frac{du}{dz} = G(u, \Psi, \sigma)$ in three-dimensional u-space for fixed Ψ and fixed $\sigma \in I$.

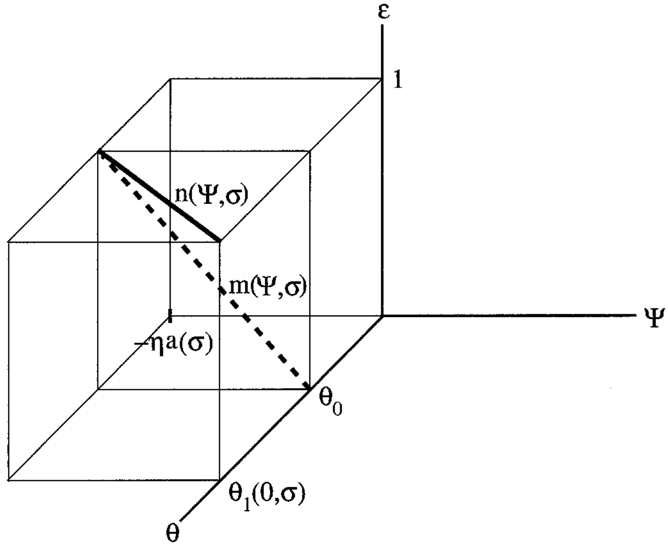


Figure 5.1: Equilibria of $\frac{du}{dz} = G(u, \Psi, \sigma)$ for fixed σ. The picture has been projected onto $\theta\epsilon\Psi$-space.

We first note that the plane $\epsilon = 1$ is invariant for each (Ψ, σ).

The point $m(\Psi, \sigma)$ has a two-dimensional stable manifold tangent at $m(\Psi, \sigma)$ to the plane $\epsilon =$ constant given by Eq. (5.4); in fact, the stable manifold is contained in this plane for $\theta \leq \theta_0$. The point $m(\Psi, \sigma)$ also has a (nonunique) one-dimensional center manifold. The flow of $\frac{du}{dz} = G(u, \Psi, \sigma)$ near $m(\Psi, \sigma)$ is determined by the flow on its center manifold.

An eigenvector for the eigenvalue 0 at $m(\Psi, \sigma)$ is $\big(X(\sigma), Y(\sigma), 1\big)$ with

$$X(\sigma) = \frac{f_\theta \eta a}{(\sigma - f_s)b} < 0, \quad Y(\sigma) = \frac{\eta a}{b} > 0. \tag{6.1}$$

Here f_s and f_θ are evaluated at (s_0, θ_0). We shall often suppress the dependence of X and Y on σ.

Since, from Eq. (4.6) and (I1), $\frac{d\epsilon}{dz} < 0$ for $\theta_0 < \theta$, the flow on the branch of the center manifold of $m(\Psi, \sigma)$ in the region $\theta_0 < \theta$ is toward $m(\Psi, \sigma)$. Thus $m(\Psi, \sigma)$ attracts nearby points that are on or above its stable manifold. See Figure 6.1.

Each point $n(\Psi, \sigma)$ has a two-dimensional stable manifold, which is an open subset of the invariant plane $\epsilon = 1$.

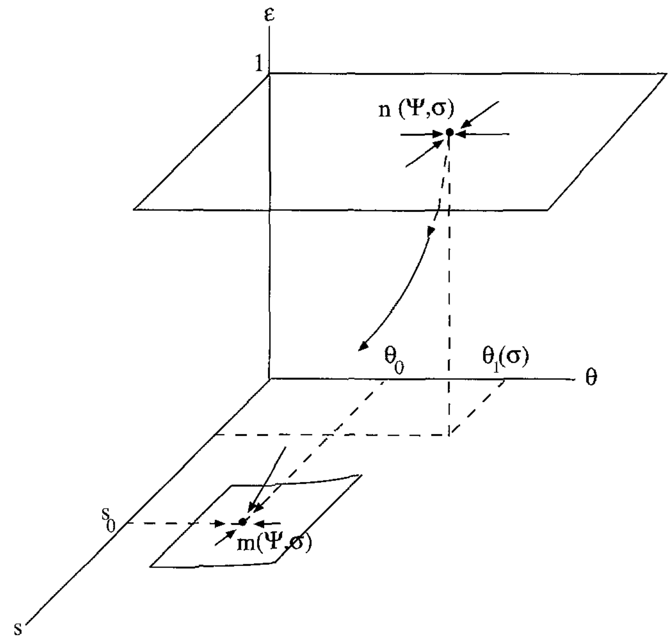


Figure 6.1: Flow of $\frac{du}{dz} = G(u, \Psi, \sigma)$ for (Ψ, σ) fixed.

For $\Psi > -\eta a(\sigma)$, the point $n(\Psi, \sigma)$ has a one-dimensional unstable manifold. Whether the unstable manifold of $n(\Psi, \sigma)$ arrives at $m(\Psi, \sigma)$, and how it does so, depends on Ψ and σ. For $\Psi = 0$ this question is studied in [8]. We shall assume

(A1) There exists σ_0 in I such that the lower branch of the unstable manifold of $n(0, \sigma_0)$ lies in the stable manifold of $m(0, \sigma_0)$.

See [8] for a discussion of the generality with which this assumption holds.

We shall further assume:

(A2) The connection of $\dot{u} = G(u, 0, \sigma_0)$ between the unstable manifold of $n(0, \sigma)$ and the stable manifold of $m(0, \sigma)$ breaks in a nondegenerate manner as σ varies.

In order to explain assumption (A2) more precisely, let $u(z) = \big(s(z), \theta(z), \epsilon(z)\big)$ be the connection of $\frac{du}{dz} = G(u, 0, \sigma_0)$ from $n(0, \sigma_0)$ to $m(0, \sigma_0)$, and let Σ be the two-dimensional plane through $u(0)$ that is perpendicular there to the connection. Let y_0 and y_1 be unit vectors in Σ, based at $u(0)$, that are respectively tangent and perpendicular to the stable manifold of $m(0, \sigma_0)$. See Figure 6.2. Write $u \in \Sigma$ as

$$u = u(0) + \alpha y_0 + \beta y_1,$$

and use (α, β) as coordinates on Σ. For (Ψ, σ) near $(0, \sigma_0)$, the unstable manifold of $n(\Psi, \sigma)$ meets Σ in a point $\big(\alpha_n(\Psi, \sigma), \beta_n(\Psi, \sigma)\big)$, with

$$\big(\alpha_n(0, \sigma_0), \beta_n(0, \sigma_0)\big) = (0, 0).$$

The stable manifold of $m(0, \sigma)$ meets Σ in a curve parameterized by, say, ξ: $\big(\alpha_m(\xi, \sigma), \beta_m(\xi, \sigma)\big)$. We may assume that $\big((\alpha_m(0, \sigma_0), \beta_m(0, \sigma_0)\big) = (0, 0)$. We must have

$$\frac{\partial \alpha_m}{\partial \xi}(0, \sigma_0) = 0, \tag{6.2}$$

and we may assume that

$$\frac{\partial \beta_m}{\partial \xi}(0, \sigma_0) \neq 0. \tag{6.3}$$

The unstable manifold of $n(0, \sigma)$ meets the stable manifold of $m(0, \sigma)$ provided there is a solution of the following system of two equations in the variables ξ and σ:

$$\alpha_n(0, \sigma) - \alpha_m(\xi, \sigma) = 0, \tag{6.4}$$

$$\beta_n(0, \sigma) - \beta_m(\xi, \sigma) = 0. \tag{6.5}$$

Proposition 6.1. *The system (6.4)–(6.5) has the regular solution* $(\xi, \sigma) = (0, \sigma_0)$ *if and only if*

$$\frac{\partial \alpha_n}{\partial \sigma}(0, \sigma_0) - \frac{\partial \alpha_m}{\partial \sigma}(0, \sigma_0) \neq 0. \tag{6.6}$$

Proof. The linearization of the system of equations in (ξ, σ) (6.4)–(6.5) at a point is given by the matrix

$$P = \begin{pmatrix} -\dfrac{\partial \alpha_m}{\partial \xi} & \dfrac{\partial \alpha_n}{\partial \sigma} - \dfrac{\partial \alpha_m}{\partial \sigma} \\ -\dfrac{\partial \beta_m}{\partial \xi} & \dfrac{\partial \beta_n}{\partial \sigma} - \dfrac{\partial \beta_m}{\partial \sigma} \end{pmatrix}.$$

At $(\xi, \sigma) = (0, \sigma_0)$, Eqs. (6.4)–(6.5) are satisfied, and by Eqs. (6.2)–(6.3), the matrix P is invertible if and only if (6.6) holds. □

The inequality (6.6) is a more precise statement of assumption (A2).

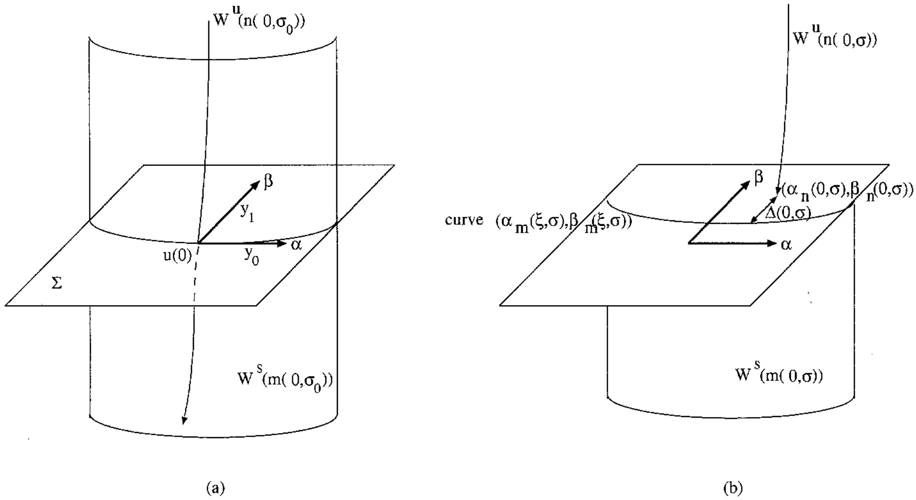


Figure 6.2: Breaking the connection: (a) $(\Psi, \sigma) = (0, \sigma_0)$, (b) $\Psi = 0$, σ near σ_0.

Remark. Let $\Delta(\Psi, \sigma)$ denote the oriented distance from the point $(\alpha_n(\Psi, \sigma), \beta_n(\Psi, \sigma))$ to the stable manifold of $m(\Psi, \sigma)$, where the distance is measured parallel to the vector y_1. It is easy to check that

$$\frac{\partial \Delta}{\partial \sigma}(0, \sigma_0) = \frac{\partial \alpha_n}{\partial \sigma}(0, \sigma_0) - \frac{\partial \alpha_m}{\partial \sigma}(0, \sigma_0).$$

In principle, $\frac{\partial \Delta}{\partial \sigma}(0, \sigma_0)$ can be computed as follows. The linear differential equation $\frac{dv}{dz} = D_u G\big(u(z), 0, \sigma_0\big)v$ has a one-dimensional space of bounded solutions, which is spanned by $\dot{u}(z)$. The adjoint equation

$$\frac{dy}{dz} = -D_u G\big(u(z), 0, \sigma_0\big)^T y$$

has a one-dimensional space of bounded solutions, which is spanned by a solution $y(z)$ with $y(0) = y_1$. Then

$$\frac{\partial \Delta}{\partial \sigma}(0, \sigma_0) = \int_{-\infty}^{\infty} y(z) \cdot \frac{\partial G}{\partial \sigma}\big(u(z), 0, \sigma_0\big)\, dz. \tag{6.7}$$

Thus (A2) is also equivalent to assuming that the Melnikov integral (6.7) is not zero. For more details, see [16].

7 Main Result

At this point we can state precisely the main result of this paper.

Theorem 7.1. *Let I be a σ-interval on which (I1)–(I4) are satisfied, and assume there is a speed $\sigma_0 \in I$ that satisfies (A1)–(A2). Then for small $\delta > 0$, the Traveling Wave System has a solution in $\mathcal{X}^4$ with speed $\sigma = \sigma_0 + O(\delta)$ that satisfies the boundary conditions (4.1) and the additional boundary condition $lim_{z\to\infty}\Psi(z) = 0$. As $z \to -\infty$, the solution approaches $\Big(m\big(\Psi^-(\delta), \sigma(\delta)\big), \Psi^-(\delta)\Big)$ with $\Psi^-(\delta) = -\eta a\big(\sigma(\delta)\big) + O(e^{-\frac{k}{\delta}})$. In particular, $\epsilon^- = 1 - O(e^{-\frac{k}{\delta}})$. As $z \to \infty$, the solution approaches $(s_0, \theta_0, 0, 0)$ exponentially at a rate that is independent of δ. As $z \to -\infty$, the solution approaches $\Big(m\big(\Psi^-(\delta), \sigma(\delta)\big), \Psi^-(\delta)\Big)$ exponentially at a rate that is $O(\delta)$.*

Remark. The expression for ϵ^- shows that not all the oxygen is burned. The remaining oxygen may be significant for larger δ.

The next five sections are devoted to completing the proof of this result.

8 Flow of $\frac{dw}{dz} = F(w, \sigma, \delta)$ for $\delta = 0$

In this section we analyze the flow of $\frac{dw}{dz} = F(w, \sigma, \delta)$, $w = (u, \Psi) = (s, \theta, \epsilon, \Psi)$, for σ near σ_0 and $\delta = 0$. The Traveling Wave System reduces to

$$\frac{du}{dz} = G(u, \Psi, \sigma), \qquad \frac{d\Psi}{dz} = 0.$$

Thus the flow is that described in the Secs. 5 and 6, except that Ψ is regarded as a state variable rather than a parameter.

We have the following structures:

(1) Let $\tilde{m}(\Psi, \sigma) = \big(m(\Psi, \sigma), \Psi\big)$ and $\tilde{n}(\Psi, \sigma) = \big(n(\Psi, \sigma), \Psi\big)$. We define the following curves of equilibria in w-space, each parameterized by Ψ:

$$M(\sigma) = \{\tilde{m}(\Psi, \sigma) : -\eta a(\sigma) \le \Psi \le 0\},$$
$$N(\sigma) = \{\tilde{n}(\Psi, \sigma) : -\eta a(\sigma) \le \Psi \le 0\}.$$

For small $\nu > 0$, we also define $N_\nu(\sigma)$ to be the subset of $N(\sigma)$ with $-\eta a(\sigma) + \nu \le \Psi \le 0$.

(2) The plane $\epsilon = 1$ in w-space is invariant under $\frac{dw}{dz} = F(w, \sigma, \delta)$ for each (σ, δ). For $\delta = 0$, within this three-dimensional plane, the curve of equilibria $N(\sigma)$ is a normally hyperbolic (in fact, attracting) manifold. For an exposition of the theory of normally hyperbolic invariant manifolds, see [15].

(3) For each $\tilde{m}(\Psi, \sigma)$ in $M(\sigma)$, define $W^s(\tilde{m}(\Psi, \sigma))$ to be the set of all (u, Ψ) such that u is in the stable manifold of $m(\Psi, \sigma)$, and define $W^c(\tilde{m}(\Psi, \sigma))$ to be the set of all (u, Ψ) such that u is in the center manifold of $m(\Psi, \sigma)$. The former is a two-dimensional manifold, and the latter is a one-dimensional manifold. Then the curve of equilibria $M(\sigma)$, regarded as a one-dimensional invariant manifold of $\frac{dv}{dz} = F(v, \sigma, 0)$, σ fixed, has a three-dimensional stable manifold $W^s(M(\sigma))$, which is the union of all $W^s(\tilde{m}(\Psi, \sigma))$ as Ψ varies, and a two-dimensional center manifold $W^c(M(\sigma))$, which is the union of all $W^c(\tilde{m}(\Psi, \sigma))$ as Ψ varies.

(4) For each $\tilde{n}(\Psi, \sigma)$ in $N(\sigma)$, define $W^s(\tilde{n}(\Psi, \sigma))$ to be the set of all (u, Ψ) such that u is in the stable manifold of $n(\Psi, \sigma)$. $W^s(\tilde{n}(\Psi, \sigma))$ is an open subset of the two-dimensional plane $\epsilon = 1$, Ψ fixed. Thus the curve of equilibria $N(\sigma)$, regarded as a one-dimensional invariant manifold of $\frac{dv}{dz} = F(v, \sigma, 0)$, σ fixed, is normally attracting within the plane $\epsilon = 1$.

(5) For each $\tilde{n}(\Psi, \sigma)$ in $N(\sigma)$ with $-\eta a(\sigma) < \Psi$ define $W^u(\tilde{n}(\Psi, \sigma))$ to be the set of all (u, Ψ) such that u is in the unstable manifold of $n(\Psi, \sigma)$. This set is a one-dimensional manifold. Then for each small $\nu > 0$, the curve of equilibria $N_\nu(\sigma)$, regarded as a one-dimensional invariant manifold of $\frac{dv}{dz} = F(v, \sigma, 0)$, σ fixed, is normally hyperbolic. It has a three-dimensional stable manifold $W^s(N_\nu(\sigma))$, which is the union of all $W^s(\tilde{n}(\Psi, \sigma))$ with $-\eta a(\sigma) + \nu \leq \Psi \leq 0$, and a two-dimensional unstable manifold $W^u(N_\nu(\sigma))$, which is the union of all $W^u(\tilde{n}(\Psi, \sigma))$ with $-\eta a(\sigma) + \nu \leq \Psi \leq 0$.

We shall now study the intersection of the two-dimensional manifold $W^u(N_\nu(\sigma))$ and the two-dimensional manifold $W^s(\tilde{m}(0, \sigma))$ in four-dimensional v-space. Notice that $W^u(N_\nu(\sigma_0))$ and $W^s(\tilde{m}(0, \sigma_0))$ meet along the connecting orbit from $\tilde{n}(0, \sigma_0)$ to $\tilde{m}(0, \sigma_0)$. One can study how the intersection breaks as σ varies by considering the intersection of each manifold with the three-dimensional plane $\tilde{\Sigma} = \Sigma \times \Psi$-space. See Figure 8.1. We continue to use (α, β) as coordinates on Σ, so that (α, β, Ψ) are coordinates on $\tilde{\Sigma}$.

The unstable manifold of $\tilde{n}(\Psi, \sigma)$ meets $\tilde{\Sigma}$ in the point $(\alpha_n(\Psi, \sigma), \beta_n(\Psi, \sigma), \Psi)$. As Ψ varies, the curve of intersection of $W^u(N_\nu(\sigma))$ with $\tilde{\Sigma}$ is swept out. The stable manifold of $\tilde{m}(0, \sigma)$ meets $\tilde{\Sigma}$ in the curve $(\alpha_m(\xi, \sigma), \beta_m(\xi, \sigma), 0)$. At $(\xi, \Psi, \sigma) = (0, 0, \sigma_0)$, the two curves meet. Other intersections can be found

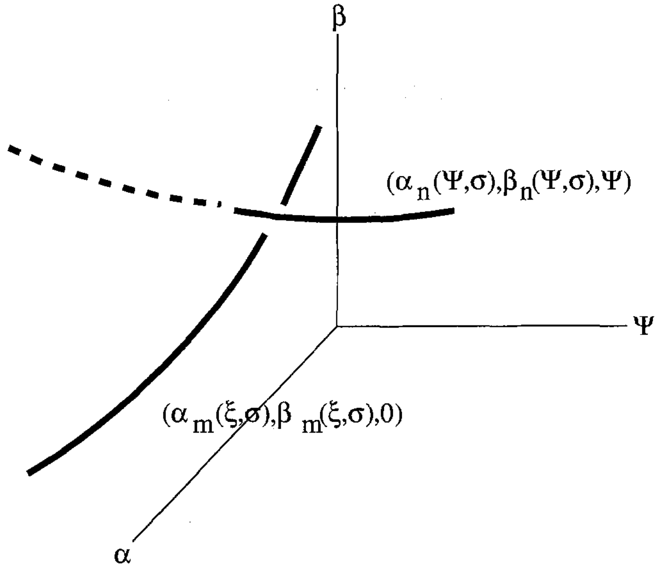


Figure 8.1: Intersection of invariant manifolds with three-dimensional $\tilde{\Sigma}$, which is pictured as $\alpha\beta$-space (Σ) crossed with Ψ-space.

by solving the following system of three equations in three unknowns:

$$\alpha_n(\Psi, \sigma) - \alpha_m(\xi, \sigma) = 0, \tag{8.1}$$

$$\beta_n(\Psi, \sigma) - \beta_m(\xi, \sigma) = 0, \tag{8.2}$$

$$\Psi = 0. \tag{8.3}$$

The linearization of this system of equations at a point is given by the matrix

$$\tilde{P} = \begin{pmatrix} -\dfrac{\partial \alpha_m}{\partial \xi} & \dfrac{\partial \alpha_n}{\partial \Psi} & \dfrac{\partial \alpha_n}{\partial \sigma} - \dfrac{\partial \alpha_m}{\partial \sigma} \\ -\dfrac{\partial \beta_m}{\partial \xi} & \dfrac{\partial \beta_n}{\partial \Psi} & \dfrac{\partial \beta_n}{\partial \sigma} - \dfrac{\partial \beta_m}{\partial \sigma} \\ 0 & 1 & 0 \end{pmatrix}.$$

At $(\xi, \Psi, \sigma) = (0, 0, \sigma_0)$, the matrix $\tilde{P}$ is invertible if and only if the matrix P defined in Section 6 is. This proves:

Proposition 8.1. *Eqs. (8.1)–(8.3) have the regular solution* $(\xi, \Psi, \sigma) = (0, 0, \sigma_0)$ *if and only if (A2) holds,* i.e., *if and only if (6.6) holds.*

9 Flow of $\frac{dw}{dz} = F(w, \sigma, \delta)$ for $\delta > 0$: Fast Connection

We now begin to analyze the flow of $\frac{dw}{dz} = F(w, \sigma, \delta)$ for σ near σ_0 and small $\delta > 0$.

(1) $M(\sigma)$ remains as a curve of equilibria. Linearization shows that for $\delta > 0$, $M(\sigma)$ has a three-dimensional stable manifold and a two-dimensional unstable manifold. The former is close to the stable manifold of $M(\sigma)$ for $\delta = 0$; the latter is close to the center manifold of $M(\sigma)$ for $\delta = 0$. (These facts follow from the Center Manifold Theorem [20].)

(2) The plane $\epsilon = 1$ remains invariant. Thus near $N(\sigma)$ there is, in the plane $\epsilon = 1$, an invariant curve $N(\sigma, \delta)$ which is hyperbolically attracting within that plane.

(3) Near $N_\nu(\sigma)$ is a normally hyperbolic invariant curve $N_\nu(\sigma, \delta)$, which can be taken to be the set of points in $N(\sigma, \delta)$ with $-\eta a(\sigma) + \nu \leq \Psi$. $N_\nu(\sigma, \delta)$ has a three-dimensional stable manifold $W^s(N_\nu(\sigma, \delta))$ and a two-dimensional unstable manifold $W^u(N_\nu(\sigma, \delta))$. The former is the closure of an open subset of the plane $\epsilon = 1$. The latter is close to $W^u(N_\nu(\sigma))$. (This fact follows from the theory of normally hyperbolic invariant manifolds [15].) From Eq. (4.8), the flow along $N_\nu(\sigma)$ is in the direction of increasing Ψ.

Proposition 9.1. *For each small $\delta \geq 0$ there is a unique speed $\sigma(\delta)$ near σ_0 such that $W^u(N_\nu(\sigma(\delta), \delta))$ contains a solution $w_\delta(z)$ that approaches $m(0, \sigma(\delta))$ as $z \to \infty$. The function $\sigma(\delta)$ is smooth, and $\sigma(0) = \sigma_0$.*

Proof. For each σ near σ_0 and small $\delta \geq 0$, the three-dimensional manifold $W^u(N_\nu(\sigma, \delta))$ meets $\tilde{\Sigma}$ in a curve $(\alpha_n(\Psi, \sigma, \delta), \beta_n(\Psi, \sigma, \delta), \Psi)$. The functions $\alpha_n(\Psi, \sigma, \delta)$ and $\beta_n(\Psi, \sigma, \delta)$ are smooth, and for $\delta = 0$ they coincide with the previously defined functions $\alpha_n(\Psi, \sigma)$ and $\beta_n(\Psi, \sigma)$.

The Center Manifold Theorem implies that for each σ near σ_0 and small $\delta \geq 0$, the three-dimensional stable manifold of $M(\sigma)$ is foliated by two-dimensional invariant surfaces $W^s(\tilde{m}(\Psi, \sigma), \delta)$, consisting of points forward asymptotic to $\tilde{m}(\Psi, \sigma)$. $W^s(\tilde{m}(\Psi, \sigma), 0)$ coincides with $W^s(\tilde{m}(\Psi, \sigma))$ defined in Sec. 8. The two-dimensional surface $W^s(\tilde{m}(0, \sigma), \delta)$ meets $\tilde{\Sigma}$ in a curve $(\alpha_m(\xi, \sigma, \delta), \beta_m(\xi, \sigma, \delta), \Psi(\xi, \sigma, \delta))$. The functions $\alpha_m(\xi, \sigma, 0)$ and $\beta_m(\xi, \sigma, 0)$ coincide with the functions $\alpha_m(\xi, \sigma)$ and $\beta_m(\xi, \sigma)$ defined earlier, and $\Psi(\xi, \sigma, 0) = 0$. Intersections of $W^u(N_\nu(\sigma, \delta))$ and $W^s(\tilde{m}(0, \sigma), \delta)$ can be found by solving the following system of three equations in four unknowns:

$$\alpha_n(\Psi, \sigma, \delta) - \alpha_m(\xi, \sigma, \delta) = 0, \tag{9.1}$$

$$\beta_n(\Psi, \sigma, \delta) - \beta_m(\xi, \sigma, \delta) = 0, \tag{9.2}$$

$$\Psi - \Psi(\xi, \sigma, \delta) = 0. \tag{9.3}$$

The linearization of Eqs. (9.1)–(9.3) at a point is given by the matrix

$$\hat{P} = \begin{pmatrix} -\frac{\partial \alpha_m}{\partial \xi} & \frac{\partial \alpha_n}{\partial \Psi} & \frac{\partial \alpha_n}{\partial \sigma} - \frac{\partial \alpha_m}{\partial \sigma} & \frac{\partial \alpha_n}{\partial \delta} - \frac{\partial \alpha_m}{\partial \delta} \\ -\frac{\partial \beta_m}{\partial \xi} & \frac{\partial \beta_n}{\partial \Psi} & \frac{\partial \beta_n}{\partial \sigma} - \frac{\partial \beta_m}{\partial \sigma} & \frac{\partial \beta_n}{\partial \delta} - \frac{\partial \beta_m}{\partial \delta} \\ -\frac{\partial \Psi}{\partial \xi} & 1 & -\frac{\partial \Psi}{\partial \sigma} & -\frac{\partial \Psi}{\partial \delta} \end{pmatrix}.$$

One solution of Eqs. (9.1)–(9.3) is $(\xi, \Psi, \sigma, \delta) = (0, 0, \sigma_0, 0)$. At this point, $\frac{\partial \Psi}{\partial \xi} = 0$ and $\frac{\partial \Psi}{\partial \sigma} = 0$, so the first 3×3 block of $\hat{P}$ equals the invertible matrix $\tilde{P}$. Therefore, by the Implicit Function Theorem, Eqs. (9.1)–(9.3) can be solved for (ξ, Ψ, σ) in terms of δ near $(\xi, \Psi, \sigma, \delta) = (0, 0, \sigma_0, 0)$; $\sigma(\delta)$ is the desired function. □

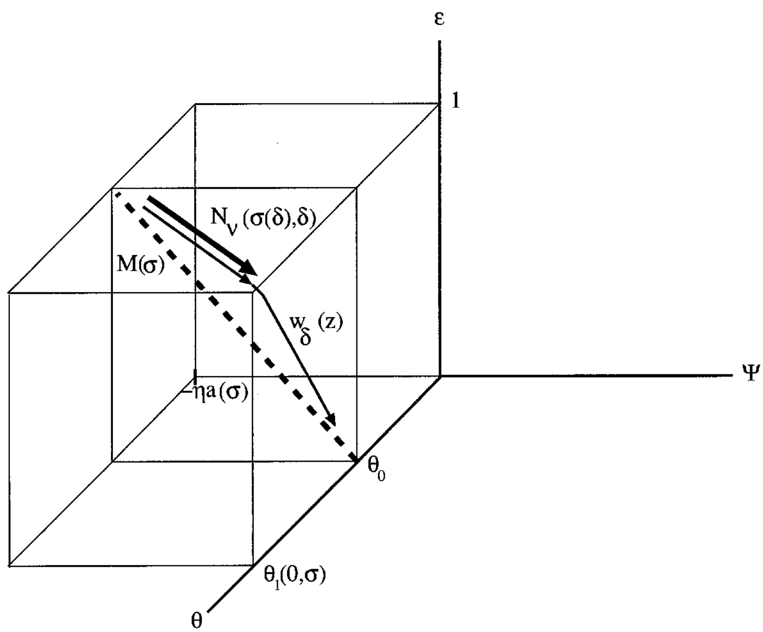


Figure 9.1: The solution $w_\delta(z)$.

Notice that for fixed (σ, δ), the curves $\big(\alpha_n(\Psi, \sigma, \delta), \beta_n(\Psi, \sigma, \delta), \Psi\big)$ and $\big(\alpha_m(\xi, \sigma, \delta), \beta_m(\xi, \sigma, \delta), \Psi(\xi, \sigma, \delta)\big)$ in $\tilde{\Sigma}$ are close to the corresponding curves pictured in Figure 6.2.

The solution $w_\delta(z)$ is shown in Figure 9.1. At $\Psi \approx 0$, in backward time, the solution quickly moves from the point $\tilde{m}\big(0, \sigma(\delta)\big)$ toward the invariant curve $N_\nu\big(\sigma(\delta), \delta\big)$. The solution then drifts slowly along $N_\nu\big(\sigma(\delta), \delta\big)$ in the direction of decreasing Ψ.

10 Flow of $\frac{dw}{dz} = F(w, \sigma, \delta)$ for $\delta > 0$: Slow Drift

As the solution $w_\delta(z)$ drifts slowly along $N_\nu\big(\sigma(\delta), \delta\big)$ in backward time, it is attracted to this curve. The result of this attraction is described by the following proposition.

Proposition 10.1. *There is a constant $k > 0$ such that for small $\delta > 0$, the solution $w_\delta(z)$ arrives at $\Psi = -\eta a(\sigma) + \nu$ within $O(e^{-\frac{k}{\delta}})$ of $N_\nu\big(\sigma(\delta), \delta\big)$.*

Proof. Let U be a small neighborhood of $N_\nu(\sigma)$ in which $\theta - \theta_0 \geq C_1 > 0$. For small $\delta > 0$, the solution $w_\delta(z)$, followed in backward time, enters U at a point with Ψ near 0. Let D be a number a little smaller that $|-\eta a(\sigma) + \nu|$. The solution $w_\delta(z)$ requires time at least $\frac{D}{C_1\delta}$ to pass along $N_\nu\big(\sigma(\delta), \delta\big)$ to $\Psi = -\eta a(\sigma) + \nu$. Once $w_\delta(z)$ is in U, there are positive constants C_2 and C_3 such that $w_\delta(z)$ approaches $N_\nu\big(\sigma(\delta), \delta\big)$ in backward time like $C_2 e^{C_3 t}$. Thus $w_\delta(z)$ arrives at $\Psi = -\eta a\big(\sigma(\delta)\big) + \nu$ within $C_2 e^{-\frac{C_3 D}{C_1 \delta}}$ of $N_\nu\big(\sigma(\delta), \delta\big)$. Notice that the constants C_i and D depend on the choice of ν and U, but are independent of δ for δ sufficiently small. Let $k = \frac{C_3 D}{C_1}$. □

11 Behavior of the connecting orbit as $z \to -\infty$

To see how the connecting orbit behaves as $z \to -\infty$, we shall use center manifold reduction.

We first make a parameter-dependent shift of coordinates in w-space. Let

$$r = s - s_0, \tag{11.1}$$

$$\omega = \theta - \theta_0, \tag{11.2}$$

$$\rho = \epsilon + \frac{\Psi}{\eta a(\sigma)}, \tag{11.3}$$

$$\Phi = \Psi + \eta a(\sigma). \tag{11.4}$$

This transformation takes

$$\{(u, \Psi, \sigma, \delta) : u = m(\Psi, \sigma)\} \tag{11.5}$$

to the subspace $r = \omega = \rho = 0$ in $(r, \omega, \rho, \Phi, \sigma, \delta)$-space, and takes

$$\{(u, \Psi, \sigma, \delta) : u = m(\Psi, \sigma), \Psi = -\eta a(\sigma)\}$$

to the subspace $r = \omega = \rho = \Phi = 0$.

Since we are studying the connecting orbits, we shall assume throughout this section that $\sigma = \sigma(\delta)$. For simplicity of presentation, we shall suppress the dependence of a and b on σ. Then in $(r, \omega, \rho, \Phi, \delta)$-coordinates, the Traveling Wave System becomes

$$\frac{dr}{dz} = \frac{1}{h(s_0 + r, \theta_0 + \omega)}\Big(a - \sigma(s_0 + r) + f(s_0 + r, \theta_0 + \omega)\Big), \tag{11.6}$$

$$\frac{d\omega}{dz} = \frac{1}{\gamma}(-b\omega + \eta a \rho), \tag{11.7}$$

$$\frac{d\rho}{dz} = \frac{1}{a}\left(\zeta(s_0 + r)\left(\rho - \frac{\Phi}{\eta a}\right)e^{-\frac{1}{\omega}} + \frac{\delta\omega}{\eta}\right), \tag{11.8}$$

$$\frac{d\Phi}{dz} = \delta\omega. \tag{11.9}$$

We add the equation

$$\frac{d\delta}{dz} = 0, \tag{11.10}$$

and we regard (11.6)–(11.10) as a five-dimensional system.

There are equilibria where $r = \omega = \rho = 0$, Φ and δ arbitrary; these correspond to the set (11.5). The plane $\rho = \frac{\Phi}{\eta a}$ is invariant. This corresponds to invariance of the plane $\epsilon = 1$ under $\frac{dw}{dz} = F(w, \sigma, \delta)$.

Let us linearize around the equilibrium at the origin. We have

$$\frac{d}{dz}\begin{pmatrix} r \\ \omega \\ \rho \\ \Phi \\ \delta \end{pmatrix} = \begin{pmatrix} \frac{f_s - \sigma}{h} & f_\theta & 0 & 0 & 0 \\ 0 & -\frac{b}{\gamma} & \frac{\eta a}{\gamma} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} r \\ \omega \\ \rho \\ \Phi \\ \delta \end{pmatrix}, \tag{11.11}$$

where f_s and f_θ are evaluated at (s_0, θ_0). Notice that the same linearization is obtained at any equilibrium with $r = \omega = \rho = \delta = 0$. The eigenvalues of this matrix are the negative numbers $\frac{f_s - \sigma}{h}$ and $-\frac{b}{\gamma}$, each with multiplicity one, and 0 with algebraic and geometric multiplicity three. A basis for the eigenspace of the eigenvalue 0 is

$$\{(X, Y, 1, 0, 0),\ (0, 0, 0, 1, 0),\ (0, 0, 0, 0, 1)\}, \tag{11.12}$$

where $X < 0$ and $Y > 0$ are given by (6.1). The three-dimensional center manifold of (11.6)–(11.10) at the origin is thus parameterized by (ρ, Φ, δ), and is given by

$$r = \rho \tilde{X}(\rho, \Phi, \delta), \tag{11.13}$$

$$\omega = \rho \tilde{Y}(\rho, \Phi, \delta), \tag{11.14}$$

with $\tilde{X}(0, \Phi, 0) = X$ and $\tilde{Y}(0, \Phi, 0) = Y$. The factor ρ in Eqs. (11.13)–(11.14) is due to the family of equilibria $r = \omega = \rho = 0$, which must lie in the center manifold. We shall suppress the dependence of $\tilde{X}$ and $\tilde{Y}$ on (ρ, Ψ, δ).

Substitution of Eqs. (11.13)–(11.14) into Eqs. (11.8)–(11.10) yields the flow on the center manifold in $\rho\Phi\delta$-coordinates, which we shall refer to as *center manifold coordinates*:

$$\frac{d\rho}{dz} = \frac{1}{a}\left(\zeta(s_0 + \rho\tilde{X})(\rho - \frac{\Phi}{\eta a})e^{-1/\rho\tilde{Y}} + \frac{\delta\rho\tilde{Y}}{\eta}\right), \tag{11.15}$$

$$\frac{d\Phi}{dz} = \delta\rho\tilde{Y}, \tag{11.16}$$

$$\frac{d\delta}{dz} = 0. \tag{11.17}$$

The plane $\rho = 0$ consists of equilibria, and the plane $\rho = \frac{\Phi}{\eta a}$ is invariant.

If we restrict to the plane $\delta = 0$, then the line $\rho = 0$ consists of equilibria with two zero eigenvalues, and the line $\rho = \frac{\Phi}{\eta a}$ also consists of equilibria. For $\Phi > 0$ these equilibria have one zero eigenvalue and one positive eigenvalue. The lines $\Phi = c$ are invariant. The flow on the two-dimensional slice of the center manifold with $\delta = 0$, near $(\rho, \Phi) = (0, 0)$, is shown in Figure 11.1(a).

The flow for fixed $\delta > 0$ is shown in Figure 11.1(b). The line $\rho = 0$ still consists of equilibria, but now one eigenvalue is 0 and the other is $\frac{\delta\tilde{Y}}{\eta a} > 0$. The line $\rho = \frac{\Phi}{\eta a}$ is now the unstable manifold of the origin. This line corresponds,

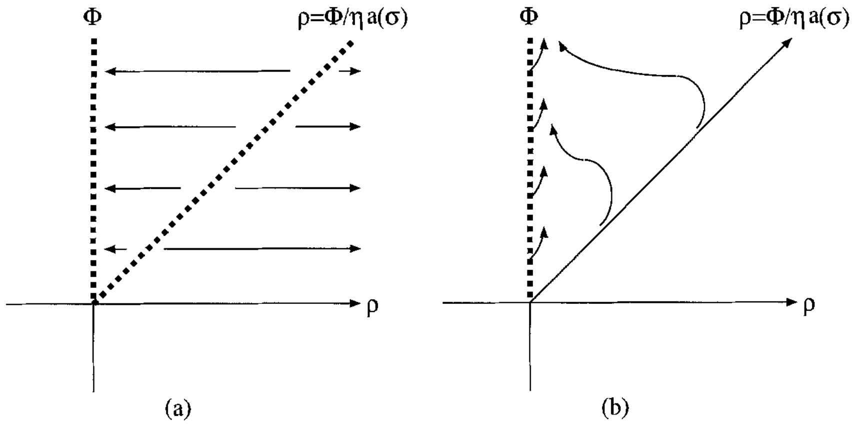


Figure 11.1: Flow on the center manifold for (a) $\delta = 0$ and (b) $\delta > 0$.

under the coordinate changes, to part of the invariant manifold $N\big(\sigma(\delta), \delta\big)$ for the Traveling Wave System. The portion of this line with $\Phi \geq \nu$ corresponds to part of $N_\nu\big(\sigma(\delta), \delta\big)$. The region $\rho > 0$, $\Phi \geq \nu$ corresponds to part of the unstable manifold of $N_\nu\big(\sigma(\delta), \delta\big)$. Notice that for $\rho > 0$, $\frac{d\Phi}{dz}$ is positive, so the flow in Figure 11.1(b) is upward.

In center manifold coordinates, the solution $w_\delta(z)$ of the Traveling Wave System that is given by Proposition 9 meets the plane $\Phi = \nu$ at the point (ρ, Φ, δ) with

$$(\rho, \Phi) = \left(\frac{\nu}{\eta a} - p(\delta), \nu\right). \tag{11.18}$$

By Proposition 10, $p(\delta)$ is $O(e^{-\frac{k}{\delta}})$.

Proposition 11.1. *As $z \to -\infty$, $w_\delta(z)$ approaches, in center manifold coordinates, a point $\big(0, \Phi_0(\delta)\big)$ with $0 < \Phi_0(\delta)$ and $\Phi_0(\delta) = O(e^{-\frac{k}{\delta}})$.*

Proof. Since $\frac{d\Phi}{dz} > 0$ and the lines $\rho = 0$ and $\rho = \frac{\Phi}{\eta a}$ are invariant, Φ decreases in backward time to a limit. Since the only invariant sets with Φ constant are points on the Φ-axis, it follows that the solution converges in backward time to a point $(0, \Phi_0)$ with $\Phi_0 > 0$.

From (11.13)–(11.14), in the region $0 < \rho < \frac{\Phi}{\eta a}$, $0 < \Phi$,

$$\frac{d\rho}{d\Phi} < \frac{\frac{1}{a}\frac{\delta\rho\tilde{Y}}{\eta}}{\delta\rho\tilde{Y}} = \frac{1}{\eta a}. \tag{11.19}$$

Now the solution of

$$\frac{d\rho}{d\Phi} = \frac{1}{\eta a} \tag{11.20}$$

that passes through the point (11.18) is the line

$$\rho = \frac{\Phi}{\eta a} - p(\delta),$$

which meets the Φ-axis at

$$\Phi_1(\delta) = \eta a p(\delta).$$

Thus $\Phi_1(\delta)$ is $O(e^{-\frac{k}{\delta}})$. The inequality (11.19) implies that $\Phi_0(\delta) \leq \Phi_1(\delta)$. □

12 Completion of the Proof of the Main Result

In this section we complete the proof of Theorem 7.1.

We have constructed a solution $w_\delta(z) = \big(s(z), \theta(z), \epsilon(z), \Psi(z)\big)$ of the Traveling Wave System with $\sigma = \sigma(\delta) = \sigma_0 + O(\delta)$. Since $w_\delta(z)$ lies in

$$W^s\Big(\tilde{m}\big(0, \sigma(\delta)\big), \delta\Big),$$

which is close to $W^s\big(\tilde{m}(0, \sigma_0), 0\big)$, $w_\delta(z)$ approaches $(s, \theta, \epsilon, \Psi) = (s_0, \theta_0, 0, 0)$ as $z \to \infty$ at an exponential rate that is independent of δ.

As $z \to -\infty$, in center manifold coordinates, $w_\delta(z)$ approaches (ρ, Φ, δ) with $(\rho, \Phi) = (0, \Phi_0(\delta))$. Using (11.1)–(11.4) and (11.13)–(11.14), we see that, as $z \to -\infty$, $w_\delta(z)$ approaches

$$(s, \theta, \epsilon, \Psi) = \left(s_0, \theta_0, 1 - \frac{\Phi_0(\delta)}{\eta a}, -\eta a + \Phi_0(\delta)\right).$$

Since the positive eigenvalue of the system (11.15)–(11.17) at an equilibrium $(0, \Phi)$ is $\frac{\delta\tilde{Y}}{\eta a}$, the desired solution of the Traveling Wave System is just $\big(s(z), \theta(z), \epsilon(z), \Psi(z)\big)$.

13 Nonexistence of the traveling wave for large heat loss

In this section we prove the following result. It states that if the rate of heat loss to the surrounding rock formation is sufficiently large, then traveling oxidation waves cannot occur.

Theorem 13.1. *For δ sufficiently large, any bounded solution of Eqs.* (4.4)–(4.6) *such that (1) $\frac{d\theta}{dz}$ is bounded and (2) $0 < s(z) < 1$ and $0 \leq \epsilon(z) \leq 1$ for all z must have $\theta(z) = \theta_0$ for all z and $\epsilon(z)$ also constant.*

To prove this result, let $\big(s(z), \theta(z), \epsilon(z)\big)$ be a bounded solution of Eqs. (4.4)–(4.6) that satisfies (1) and (2). We shall regard $s(z)$ and $\epsilon(z)$ as given, and we shall show that it must be the case that $\theta(z) = \theta_0$ for all z. Then from Eq. (4.6) we see that $\epsilon(z)$ is also constant, so the theorem is proved.

Let $\omega = \theta - \theta_0$. From Eqs. (4.5)–(4.6) and the definitions of b and q, we have

$$\frac{d^2\omega}{dz^2} = -\frac{b}{\gamma}\frac{d\omega}{dz} + \frac{\delta}{\gamma}\omega + N(z, \omega), \tag{13.1}$$

where

$$N(z, \omega) = \begin{cases} -\frac{\eta\zeta}{\gamma} s(z)\big(1 - \epsilon(z)\big)e^{-\frac{1}{\omega}} & \text{if } \omega > 0, \\ 0 & \text{if } \omega \leq 0. \end{cases} \tag{13.2}$$

(Recall that $s(z)$ and $\epsilon(z)$ are given.)

The function $\frac{e^{-\frac{1}{\omega}}}{\omega^2}$, which is the derivative of $e^{-\frac{1}{\omega}}$, has, on the interval $0 < \omega < \infty$, a maximum value of

$$M = \frac{4}{e^2} \approx .54$$

at $\omega = \frac{1}{2}$. Therefore

$$|N(z, \omega) - N(z, \tilde{\omega})| \leq \frac{\eta\zeta M}{\gamma}|\omega - \tilde{\omega}|. \tag{13.3}$$

Let $y = (y_1, y_2) = (\omega, \frac{d\omega}{dz})$. Then Eq. (13.1) is equivalent to the system

$$\frac{dy}{dz} = Ay + \big(0, N(z, y_1)\big) \tag{13.4}$$

with

$$A = \begin{pmatrix} 0 & 1 \\ \frac{\delta}{\gamma} & -\frac{b}{\gamma\theta_0} \end{pmatrix}. \tag{13.5}$$

The eigenvalues of A are

$$\lambda_{\pm} = \frac{1}{2\gamma}\left(-b \pm \sqrt{b^2 + 4\gamma\delta}\right). \tag{13.6}$$

We shall assume that $b > 0$, so that $\lambda_+ > 0$ and $\lambda_- < 0$; the cases $b < 0$ and $b = 0$ are similar. Corresponding eigenvectors are $(1, \lambda_{\pm})$. Let

$$\Lambda = \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1 \\ \lambda_- & \lambda_+ \end{pmatrix}, \quad U^{-1} = \frac{1}{\lambda_+ - \lambda_-}\begin{pmatrix} \lambda_+ & -1 \\ -\lambda_- & 1 \end{pmatrix}, \tag{13.7}$$

so that $U^{-1}AU = \Lambda$. Notice that A, $\lambda_{\pm}$, Λ, and U are all functions of σ and δ. Let $y = Ux$. Then Eq. (13.4) becomes

$$\frac{dx}{dz} = \Lambda x + P(z, x, \sigma, \delta), \tag{13.8}$$

where

$$P(z, x_1, x_2, \sigma, \delta) = U^{-1}\big(0, N(z, x_1 + x_2)\big) = \frac{1}{\lambda_+ - \lambda_-} N(z, x_1 + x_2)(-1, 1).$$

Let $C(\Re, \Re^{\ell})$ denote the Banach space of bounded continuous functions from $\Re$ to $\Re^{\ell}$, $\ell = 1,\ 2$. In $C(\Re, \Re)$ the norm is $\|k\| = \sup(|k(z)| : z \in \Re)$. In $\Re^2$ we use the norm $\|(x_1, x_2)\| = \max(|x_1|, |x_2|)$, and we use the corresponding norm in $C(\Re, \Re^2)$:

$$\begin{aligned} \|x\| = \sup(\|x(z)\| : z \in \Re) &= \sup\big(\|(x_1(z), x_2(z)\|\big) : z \in \Re) \\ &= \max(\|x_1\|, \|x_2\|). \end{aligned}$$

The following lemma is an easy consequence of the Variation of Constants formula.

Lemma 13.2. *Let* $h(z) = \big(h_1(z), h_2(z)\big) \in C(\Re, \Re^2)$. *Then the only bounded solution of* $\frac{dx}{dz} = \Lambda x + h$ *is* $x(z) = \big(x_1(z), x_2(z)\big)$ *with*

$$x_1(z) = \int_{-\infty}^{z} e^{\lambda_-(z-s)} h_1(s)\, ds, \qquad x_2(z) = \int_{\infty}^{z} e^{\lambda_+(z-s)} h_2(s)\, ds.$$

In addition, $|x_1| \le -\frac{1}{\lambda_-}|h_1|$ *and* $|x_2| \le \frac{1}{\lambda_+}|h_2|$.

Using Lemma 13.2, we define a linear mapping L from $C(\Re, \Re^2)$ to itself by $Lh = x$. We also define a mapping

$$\hat{P} : C(\Re, \Re^2) \times \Re \times \Re_+ \to C(\Re, \Re^2)$$

by $\hat{P}(x,\sigma,\delta)(z) = P\big(z, x(z), \sigma, \delta\big)$. From Eq. (13.8), if, for some (σ,δ), $\big(s(z), \theta(z), \epsilon(z)\big)$ is a bounded solution of Eqs. (4.4)–(4.6) that satisfies (1) and (2), and $x(z)$ is related to $\big(s(z), \theta(z), \epsilon(z)\big)$ as described in this section, then we must have $x = L\hat{P}(x,\sigma,\delta)$, *i.e.*, x must be a fixed point of the mapping $L\hat{P}(\cdot,\sigma,\delta)$.

Let $P_i(z,x,\sigma,\delta)$ denote the ith component of $P(z,x,\sigma,\delta)$, $i = 1,\ 2$. We have, for each $i = 1,\ 2$, for each z, and for each x and $\tilde{x}$ in $\Re^2$,

$$\begin{aligned}|P_i(z,x,\sigma,\delta) - P_i(z,\tilde{x},\sigma,\delta)| &\le \frac{1}{\lambda_+ - \lambda_-}|N(z, x_1 + x_2) - N(z, \tilde{x}_1 + \tilde{x}_2)| \\ &\le \frac{\eta\zeta M}{\gamma(\lambda_+ - \lambda_-)}|x_1 + x_2 - \tilde{x}_1 - \tilde{x}_2| \\ &\le \frac{\eta\zeta M}{\gamma(\lambda_+ - \lambda_-)}(|x_1 - \tilde{x}_1| + |x_2 - \tilde{x}_2|) \\ &\le \frac{2\eta\zeta M}{\gamma(\lambda_+ - \lambda_-)}\|x - \tilde{x}\| = \frac{2\eta\zeta M}{\sqrt{b^2 + 4\gamma\delta}}\|x - \tilde{x}\|.\end{aligned}$$

Therefore, if x and $\tilde{x}$ are in $C(\Re, \Re^2)$,

$$\|\hat{P}(x,\sigma,\delta) - \hat{P}(\tilde{x},\sigma,\delta)\| \le \frac{2\eta\zeta M}{\sqrt{b^2 + 4\gamma\delta}}\|x - \tilde{x}\|.$$

Since $0 < \lambda_+ < -\lambda_-$, Lemma 13.2 implies that $\|L\| = \frac{1}{\lambda_+}$. Therefore

$$\|L\hat{P}(x,\sigma,\delta) - L\hat{P}(\tilde{x},\sigma,\delta)\| \le \frac{2\eta\zeta M}{\lambda_+\sqrt{b^2 + 4\gamma\delta}}\|x - \tilde{x}\|.$$

Now

$$\begin{aligned}\frac{2\eta\zeta M}{\lambda_+\sqrt{b^2 + 4\gamma\delta}} &= \frac{4\eta\zeta\gamma M}{\big(-b + \sqrt{b^2 + 4\gamma\delta}\big)\sqrt{b^2 + 4\gamma\delta}} \\ &= \frac{\eta\zeta M}{\delta}\left(\frac{b}{\sqrt{b^2 + 4\gamma\delta}} + 1\right) \le \frac{2\eta\zeta M}{\delta}. \qquad (13.9)\end{aligned}$$

Therefore, if

$$\delta > 2\eta\zeta M, \qquad (13.10)$$

then $L\hat{P}(\cdot,\sigma,\delta)$ is a contraction of $C(\Re, \Re^2)$ for each (σ,δ). Notice that the inequality (13.10) is independent of σ.

Now $N(z, 0) = 0$ for all z and σ, so $P(z, 0, \sigma, \delta) = 0$ for all z, σ, and δ. Therefore $x = 0$ is a fixed point of $L\hat{P}(\cdot, \sigma, \delta)$ for each (σ, δ). For δ sufficiently large, $L\hat{P}(\cdot, \sigma, \delta)$ is a contraction, so $x = 0$ is the only fixed point. Since $x(z) = 0$ implies $\theta(z) = \theta_0$, the result follows.

Remark. Figure 2.1 illustrates a traveling wave with $\delta = .003334541$. Since $\eta = 5$ and $\zeta = 1$ in that example, the estimate (13.10) implies that no traveling wave exists for $\delta > 10M \approx 5.4$. In fact, if we attempt to continue the traveling wave in δ, we find that σ decreases, and there is a turning point at δ about 0.0178.

14 Conclusions and Discussion

In this work we have considered the existence of oxidation heat pulses excited in a petroleum reservoir originally under oxygen or air injection, so that a uniform ratio of oil to oxygen is in place initially. We have shown (see the end of Sec. 10) that the width of the slow cooling part of the pulse increases unboundedly as the heat loss to the surrounding rock formation decreases. This is the case, for example, when the thickness of the petroleum-bearing formation increases, or when the total seepage velocity of the fluid increases. When heat loss is very small, only the lead front of the pulse may fit between the injection and the producing wells. When the heat loss vanishes, the triangular oxidation pulse reduces to an oxidation bank, with no decay behind it.

On the other hand, in the case of excessive heat loss to the surroundings, we have shown that no oxidation pulses are supported by the medium. An important open problem is to understand how the pulse vanishes as the heat loss increases.

Our analysis implies that if simulations of combustion processes in petroleum engineering are to predict correctly the occurrence of combustion pulses, they must take into account heat loss to the rock formation.

We have treated a severely simplified model in order to avoid complications in the analysis and thereby focus on the essential mathematical issues. In a companion paper [17], the focus is the modeling, and most unphysical simplifications are removed. After an analysis that is not harder but more complicated, we arrive at physical conclusions that are basically the same. Taken together, these papers suggest that the simplified model analyzed in this work captures the essential features of combustion of fluids in porous media.

Appendix A. Nomenclature

ϕ – rock porosity
ρ_g, ρ_o – densities of gaseous and oleic phase, assumed not to depend on pressure or temperature
ρ_r – density of rock
s_g, s_o – saturations of gaseous and oleic phase
v_g, v_o – seepage velocities of gaseous and oleic phase
C_g, C_o, C_r – heat capacities of gaseous, oleic and solid phase at constant pressure
κ_g, κ_o, κ_r – heat conductivities of gaseous, oleic and solid phase
θ – temperature (assumed to be the same in gaseous, oleic and solid phase)
p_{ij} – capillary pressure:pressure difference between phase i and j
λ_i – mobility of phase i
k_i – relative permeability of phase i
μ_i – viscosity of phase i
p_i – pressure of phase i
v_i – seepage velocity of phase i
f_i – fractional flow of phase i
E_i – massic energy density of phase i per unit volume
K – absolute permeability of rock
λ – total mobility of fluid
v – total seepage velocity
q – rate of oxygen consumption
ϵ – burned volume fraction of the gaseous phase

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